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Giant Gravitons and Fuzzy CP^2

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ABSTRACT

In this article we describe the giant graviton configurations in $AdS_m \times S^n$ backgrounds that involve 5-spheres, namely, the giant graviton in $AdS_4 \times S^7$ and the dual giant graviton in $AdS_7 \times S^4$, in terms of dielectric gravitational waves. Thus, we conclude the programme initiated in hep-th/0207199 and pursued in hep-th/0303183 and hep-th/0406148 towards the microscopical description of giant gravitons in $AdS_m \times S^n$ spacetimes. In our construction the gravitational waves expand due to Myers dielectric effect onto “fuzzy 5-spheres” which are described as S^1 bundles over fuzzy CP^2 . These fuzzy manifolds appear as solutions of the matrix model that comes up as the action for M-theory gravitational waves. The validity of our description is checked by confirming the agreement with the Abelian description in terms of a spherical M5-brane when the number of waves goes to infinity.

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1 Introduction

It is by now well-known that giant gravitons can be described microscopically in terms of dielectric [1] gravitational waves [2, 3, 4, 5, 6]. In $AdS_m \times S^n$ spacetimes the gravitational waves expand into a fuzzy $S^{(n-2)}$ brane included in S^n , for the genuine giant graviton, or into a fuzzy $S^{(m-2)}$ brane included in AdS_m , for the dual giant graviton, both carrying angular momentum on the spherical part of the geometry. For the giant graviton in $AdS_7 \times S^4$ and the dual giant graviton in $AdS_4 \times S^7$ these fuzzy spheres are ordinary non-commutative S^2 [4], whereas for both the giant and dual giant gravitons in $AdS_5 \times S^5$ the corresponding ‘‘fuzzy 3-spheres’’ are defined as S^1 bundles over non-commutative S^2 base manifolds [5]. In all cases perfect agreement is found, when the number of waves goes to infinity, with the (Abelian) description in [7, 8, 9] in terms of spherical test branes. This agreement provides the strongest support to the (non-Abelian) dielectric constructions of [4, 5].

The key point in the non-Abelian description of giant gravitons is the construction of the action for the system of coincident waves, and the identification of the dielectric and magnetic moment couplings responsible of their expansion. Given that $AdS_m \times S^n$ is not weakly curved one cannot use linearised Matrix theory results as those in [10, 11].

The action appropriate to describe coincident gravitational waves in non-weakly curved M-theory backgrounds was constructed in [4]. This action contains dielectric and magnetic moment couplings to the 3-form potential of eleven dimensional supergravity, which are responsible for the expansion of the waves into a dielectric or magnetic moment M2-brane with the topology of a fuzzy S^2 , which constitute, respectively, the dual giant and giant graviton configurations in the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds. Using this action one can also describe coincident Type IIB gravitational waves in non-weakly curved AdS type backgrounds, after reduction and T-duality [5]. In the Type IIB action the T-duality direction occurs as an special isometric direction, and this turns out to be essential in the construction of the fuzzy manifold associated to both giant and dual giant graviton configurations in $AdS_5 \times S^5$. Describing the ‘‘fuzzy 3-sphere’’ as an S^1 bundle over a fuzzy 2-sphere, the isometric direction is precisely the coordinate along the fibre. This same action was later used in [6] to describe the giant graviton solutions in the $AdS_3 \times S^3 \times T^4$ Type IIB background in terms of expanding waves. In this case the waves expand into a fuzzy cylinder whose basis is contained either in S^3 or in AdS_3 . In all these cases the agreement between these descriptions and the Abelian descriptions of [7, 8, 9] for large number of waves provides the strongest support for the validity of the non-Abelian actions for coincident waves⁴.

In this paper we would like to complete the programme initiated in [4], and pursued in [5] and [6], with the microscopical description of the giant graviton in $AdS_4 \times S^7$ and the dual giant graviton in $AdS_7 \times S^4$. One expects that microscopically these gravitons will be described in terms of a magnetic moment or dielectric M5-brane with the topology of a fuzzy 5-sphere. Fuzzy S^n with $n > 2$ are however quite complicated technically. The general strategy is to identify them as subspaces of suitable spaces (spaces which admit a symplectic structure) and to introduce conditions to restrict the functions to be on the sphere [12, 13] [14]. We will see that in our construction the ‘‘fuzzy S^5 ’’ is simply defined as an S^1 bundle over a fuzzy CP^2 , in very much the same way the ‘‘fuzzy S^3 ’’ in [5] was defined as an S^1 bundle over a fuzzy S^2 . Again the coordinate along the S^1 fibre is, as in the fuzzy S^3 of [5], an

⁴Of course, together with the fact that the action for M-theory waves reduces to Myers action for D0-branes when they propagate along the eleventh direction. This was in fact the key ingredient in [4] in order to extend the linearised Matrix theory result to more general backgrounds. See this reference for more details.

isometric direction present in the action describing the waves. The existence of this direction is, moreover, crucial in order to construct the right dielectric couplings that will cause the expansion of the waves. As we will discuss the expanded 5-brane will be a longitudinal brane wrapped on this direction.

The fuzzy CP^2 has been extensively studied in the literature (see for instance [15]-[22]). In the context of Myers dielectric effect it was studied in [23] [24]. CP^2 is the coset manifold $SU(3)/U(2)$. G/H coset manifolds can be described as fuzzy surfaces if H is the isotropy group of the lowest weight state of a given irreducible representation of G [25, 23]. When $G = SU(2)$ all lowest weight vectors have isotropy group $U(1)$, and therefore one describes a fuzzy $SU(2)/U(1)$, i.e. a fuzzy S^2 for any choice of irreducible representation. In the case of $SU(3)$ irreducible representations can be parametrised by two integers (n, m) , corresponding to the number of fundamental and anti-fundamental indices. The lowest weight vector in the representations $(n, 0)$ or $(0, n)$ has isotropy group $U(2)$, whereas for any other irreducible representation it has isotropy group $U(1) \times U(1)$. Therefore, choosing a $(n, 0)$ or a $(0, n)$ irreducible representation one describes a fuzzy CP^2 .

We will use this result to describe “the fuzzy S^5 ”, associated to the giant graviton in $AdS_4 \times S^7$ and the dual giant graviton in $AdS_7 \times S^4$, as an S^1 bundle over a fuzzy CP^2 base manifold. We will then use the action for coincident M-theory gravitational waves to find the corresponding ground state configuration. A key ingredient in this construction is the identification in the action of the direction along the S^1 bundle. For this purpose we will start in Section 2 by recalling some properties of the action for coincident M-theory gravitational waves constructed in [4]. Then in Section 3 we will use this action to describe microscopically the giant graviton in $AdS_4 \times S^7$. We will see that the corresponding macroscopical description is in terms of a longitudinal M5-brane with S^5 topology. In Section 4 we present the analogous description for the dual giant graviton in $AdS_7 \times S^4$. In both cases we show the explicit agreement with the macroscopical description in [7, 8] for large number of gravitons. In Section 5 we present our Conclusions, where we discuss the connection between our solution and other 5-brane solutions to Matrix theory actions previously found in the literature, as well as the supersymmetry properties of our configurations.

2 The action for M-theory gravitational waves

The action for coincident M-theory gravitational waves constructed in [4] is given by:

$$S = S^{\text{BI}} + S^{\text{CS}} \quad (2.1)$$

with BI action given by

$$S^{\text{BI}} = -T_0 \int d\tau \text{STr}\{k^{-1} \sqrt{-P[E_{00} + E_{0i}(Q^{-1} - \delta)_k^i E^{kj} E_{j0}] \det Q}\}, \quad (2.2)$$

where

$$E_{\mu\nu} = \mathcal{G}_{\mu\nu} + k^{-1} (i_k C^{(3)})_{\mu\nu}, \quad \mathcal{G}_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \quad (2.3)$$

and

$$Q_j^i = \delta_j^i + ik[X^i, X^k]E_{kj}, \quad (2.4)$$

and CS action given by

$$S^{\text{CS}} = T_0 \int d\tau \text{STr} \left\{ -P[k^{-2}k^{(1)}] + iP[(i_X i_X)C^{(3)}] + \frac{1}{2}P[(i_X i_X)^2 i_k C^{(6)}] - \frac{i}{6}P[(i_X i_X)^3 i_k N^{(8)}] \right\} \quad (2.5)$$

where $i_k N^{(8)}$ denotes the Kaluza-Klein monopole potential [26]. In this action k^μ is an Abelian Killing vector that points on the direction of propagation of the waves. This direction is isometric, because the background fields are either contracted with the Killing vector, so that any component along the isometric direction of the contracted field vanishes, or pulled back in the worldvolume with covariant derivatives relative to the isometry (see [4] for their explicit definition)⁵. To understand why this is so we need to recall the construction of this action.

Expression (2.1) was obtained by uplifting to eleven dimensions the action for Type IIA gravitational waves derived in [27] using Matrix String theory in a weakly curved background, and then going beyond the weakly curved background approximation by demanding agreement with Myers action for D0-branes when the waves propagate along the eleventh direction.

In the action for Type IIA waves the circle in which one Matrix theory is compactified in order to construct Matrix String theory cannot be decompactified in the non-Abelian case [27]. In fact, the action exhibits a $U(1)$ isometry associated to translations along this direction, which by construction is also the direction on which the waves propagate. A simple way to see this is to recall that the last operation in the 9-11 flip involved in the construction of Matrix String theory is a T-duality from fundamental strings wound around the 9th direction. Accordingly, in the action we find a minimal coupling to $g_{\mu 9}/g_{99}$, which is the momentum operator $k^{-2}k_\mu$ in adapted coordinates. Therefore, by construction, the action (2.1) is designed to describe BPS waves with momentum charge along the compact isometric direction. It is important to mention that in the Abelian limit, when all dielectric couplings and $U(N)$ covariant derivatives⁶ disappear, (2.1) can be Legendre transformed into an action in which the dependence on the isometric direction has been restored. This action is precisely the usual action for a massless particle written in terms of an auxiliary γ metric (see [4] and [27] for the details), where no information remains about the momentum charge carried by the particle.

Let us now look at the couplings to the 3-form potential of eleven dimensional supergravity. We clearly find a dipole coupling in the CS part of the action and a magnetic moment coupling

$$[X^i, X^k](i_k C^{(3)})_{kj} \quad (2.6)$$

in the BI part⁷. These couplings play a crucial role in the microscopical description of the $AdS_4 \times S^7$ dual giant graviton and the $AdS_7 \times S^4$ giant graviton, respectively.

Let us parametrise the $AdS_m \times S^n$ background as

$$ds^2 = -(1 + \frac{r^2}{\tilde{L}^2})dt^2 + \frac{dr^2}{1 + \frac{r^2}{\tilde{L}^2}} + r^2 d\Omega_{m-2}^2 + L^2(d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\Omega_{n-2}^2) \quad (2.7)$$

$$C_{t\alpha_1 \dots \alpha_{m-2}}^{(m-1)} = -\frac{r^{m-1}}{\tilde{L}} \sqrt{g_\alpha}, \quad C_{\phi\beta_1 \dots \beta_{n-2}}^{(n-1)} = a_n L^{n-1} \sin^{n-1} \theta \sqrt{g_\beta} \quad (2.8)$$

⁵The reduced metric $\mathcal{G}_{\mu\nu}$ appearing in (2.3) is in fact defined such that its pull-back with ordinary derivatives equals the pull-back of $g_{\mu\nu}$ with these covariant derivatives.

⁶Which are of course implicit in the pull-backs of the non-Abelian action (2.1).

⁷Let us stress that through its contraction with the Killing vector the 3-form potential acquires the necessary rank to couple in the BI action.

where $a_4 = a_5 = 1$, $a_7 = -1$, α_i (β_i) parametrise the S^{m-2} (S^{n-2}) contained in AdS_m (S^n) as

$$d\Omega_{m-2}^2 = d\alpha_1^2 + \sin^2 \alpha_1 (d\alpha_2^2 + \sin^2 \alpha_2 (\dots + \sin^2 \alpha_{m-3} d\alpha_{m-2}^2)) \quad (2.9)$$

(similarly for β_i) and $\sqrt{g_\alpha}$ ($\sqrt{g_\beta}$) denotes the volume element on the unit S^{m-2} (S^{n-2}).

Consider the $AdS_7 \times S^4$ background and take $r = 0$, $\theta = \text{constant}$ and ϕ time-dependent, i.e. the ansatz for the giant graviton configuration. In this case there is a non-vanishing 3-form potential

$$C_{\phi\beta_1\beta_2}^{(3)} = L^3 \sin^3 \theta \sqrt{g_\beta} \quad (2.10)$$

which (when rewritten in terms of Cartesian coordinates) clearly couples as in (2.6), given that the gravitons carry P_ϕ angular momentum, which identifies ϕ with the isometric direction in the action (so that $k^\mu = \delta_\phi^\mu$). On the other hand, taking the $AdS_4 \times S^7$ background and the dual giant graviton ansatz, $\theta = 0$, $r = \text{constant}$ and ϕ time-dependent, the non-vanishing 3-form potential is

$$C_{t\alpha_1\alpha_2}^{(3)} = -\frac{r^3}{\tilde{L}} \sqrt{g_\alpha} \quad (2.11)$$

which couples through (2.5). The detailed computations of the potentials associated to these configurations were performed in [4], and perfect agreement was found for large number of gravitons with the macroscopical calculations in [7, 8].

Consider now the backgrounds in which the gravitons expand into M5-branes. Taking the giant graviton ansatz in the $AdS_4 \times S^7$ background one finds a non-vanishing 6-form potential

$$C_{\phi\beta_1\dots\beta_5}^{(6)} = -L^6 \sin^6 \theta \sqrt{g_\beta}. \quad (2.12)$$

Clearly this potential does not couple in the action for the system of waves if we identify ϕ with the isometric direction, given that in the pull-back involved in

$$\int d\tau P[(i_X i_X)^2 i_k C^{(6)}] \quad (2.13)$$

only ϕ is time-dependent, and this component is already taken through the interior product with k^μ . Similarly, the dual giant graviton ansatz in $AdS_7 \times S^4$ yields

$$C_{t\alpha_1\dots\alpha_5}^{(6)} = -\frac{r^6}{\tilde{L}} \sqrt{g_\alpha} \quad (2.14)$$

which again does not couple in the action, this time because the quadrupolar coupling (2.13) has ϕ component and therefore is not of the form (2.14).

A puzzle then arises regarding the microscopical description of these giant graviton configurations in terms of dielectric gravitational waves. By analogy with other expanded configurations one would expect the gravitons to expand into fuzzy 5-spheres due to quadrupolar electric or magnetic moment couplings to the 6-form potential. Since a 5-sphere has 5 relative dimensions with respect to a point-like object, the 6-form potential has to be contracted as well with the Killing direction in order to be able to couple to a one dimensional worldvolume. However, we have seen that if this Killing direction is the direction of propagation the only coupling of this form present in the action for M-theory waves (2.1) vanishes for the AdS backgrounds that we want to study.

In order to find a possible solution to this puzzle let us recall the microscopical description of the giant graviton configurations in the $AdS_5 \times S^5$ background [5]. These configurations correspond microscopically to Type IIB gravitational waves expanding into S^3 D3-branes due to their dielectric or magnetic moment interaction with the 4-form RR potential of the background. This potential is

$$C_{\phi\beta_1\beta_2\beta_3}^{(4)} = L^4 \sin^4 \theta \sqrt{g_\beta}, \quad (2.15)$$

for the giant graviton, and

$$C_{t\alpha_1\alpha_2\alpha_3}^{(4)} = -\frac{r^4}{L} \sqrt{g_\alpha} \quad (2.16)$$

for the dual giant graviton. Consider now that the action for Type IIB waves contained the coupling that one would naturally expect to involve the 4-form potential:

$$\int d\tau P[(i_X i_X) i_k C^{(4)}]. \quad (2.17)$$

By the same arguments above one easily checks that this coupling vanishes both for the giant and dual giant graviton potentials.

The action describing Type IIB waves constructed in [5] contains however a second isometric direction, with Killing vector l^μ . This direction is the direction in which one performs the T-duality transformation that has to be made in order to obtain the action for Type IIB waves from the (reduction of the) action for M-theory waves. In the Abelian limit one can interpret the resulting action as a dimensional reduction over the T-duality direction, as one usually does, but this cannot be done in the non-Abelian case, in part due to the presence of non-trivial dielectric couplings. One finds, in particular, the following coupling in the CS part of the action

$$\int d\tau P[(i_X i_X) i_l C^{(4)}], \quad (2.18)$$

together with a magnetic moment coupling to the 4-form potential in the BI part of the action,

$$[X^i, X^k](i_k i_l C^{(4)})_{kj}. \quad (2.19)$$

This isometric action for Type IIB waves is therefore valid to describe waves propagating in backgrounds which contain a $U(1)$ isometric direction. This is the case for the $AdS_5 \times S^5$ background, where the $U(1)$ isometry is that corresponding to the translations along the S^1 -fibre in the description of the 3-sphere (contained in S^5 (for the giant graviton) or AdS_5 (for the dual giant graviton)) as an S^1 -fibre over an S^2 base manifold. In fact rewriting the potentials (2.15) and (2.16) in adapted coordinates to this isometry it is easy to see that the coupling (2.18) is non-vanishing for the 4-form potential associated to the dual giant graviton and the coupling (2.19) for the one associated to the giant graviton. Indeed the detailed computation of the corresponding non-Abelian potentials shows perfect agreement with the macroscopical calculations in [7, 9] for large number of gravitons. Let us stress however that the right dielectric couplings that cause the expansion of the gravitons can only be constructed in spacetimes with a $U(1)$ isometry.

The discussion above suggests that something similar can be happening for the gravitons expanding into 5-spheres that we are considering in this article. The S^5 can similarly be

described as a $U(1)$ bundle, in this case over the two dimensional complex projective space, CP^2 . Therefore, there is a $U(1)$ isometry in the background that could allow the construction of further dielectric couplings.

However, the action that we know for M-theory waves contains only one isometric direction, that we naturally identified with the direction of propagation of the waves, since, as we discussed, they are minimally coupled to the momentum operator in this direction. Consider instead that we identified this direction with the $U(1)$ fibre of the $U(1)$ -decomposition of the 5-sphere. In this case one can see that the coupling (2.13) is non-vanishing both for the giant and dual giant graviton potentials. We will see this in detail in the next sections, when we write S^5 as an S^1 fibre over CP^2 in adapted coordinates. Denoting χ the coordinate adapted to the isometry, (2.13) becomes

$$\int d\tau X^l X^k X^j X^i \left[C_{\chi i j k l 0}^{(6)} + C_{\chi i j k l \phi}^{(6)} \dot{\phi} \right], \quad (2.20)$$

and one easily finds that the first term is non-vanishing for the dual giant graviton whereas the second one is non-vanishing for the giant graviton.

We then propose to use the action (2.1) to describe gravitons propagating in a spacetime with a compact isometric direction, χ , with, by construction, non-vanishing momentum-charge along that direction, P_χ , and with a non-zero velocity along a different transverse direction, $\dot{\phi}$, as implied by the giant and dual giant gravitons ansätze. Clearly, in order to describe giant graviton configurations, which only carry momentum P_ϕ , we will have to set the momentum charge P_χ to zero at the end of the calculation. We will see that this calculation matches exactly the macroscopical calculation in [7, 8] for large number of gravitons, which will provide the strongest check for the validity of our proposal.

Of course, a more direct microscopical description of giant gravitons with momentum P_ϕ living in a spacetime with an isometric direction χ would be in terms of an effective action containing both χ and ϕ as isometric directions, and with momentum charge only with respect to the second one. As we have seen this type of action exists in the Type IIB theory. However the construction of a similar type of action for M-theory waves cannot be made based on duality arguments. One possibility would be to start from the two isometric action for Type IIB waves, T-dualize and uplift to eleven dimensions. The detailed calculation shows that in the non-Abelian case both the T-duality direction and the eleventh dimension become isometric directions in the M-theory action. Therefore, the resulting action is adequate for the study of M-theory gravitational waves propagating in a spacetime with three $U(1)$ directions (the other isometric direction is the direction of propagation of the waves). These isometries are however not present in the M-theory backgrounds that we want to study.

Let us finally remark that an action for M-theory gravitational waves with two isometric directions would have to be highly non-perturbative. This would be in the same spirit of [3], where it is argued that the 5-brane cannot appear as a classical solution to the pp-wave Matrix model because the scaling of its radius with the coupling constant is more non-perturbative than the one corresponding to a classical solution. Coming back to our action, if the direction of propagation is also isometric, we cannot have a magnetic coupling to the 6-form potential like the second one in (2.20). Therefore the only way to find such a magnetic coupling is in the BI action, through something like

$$Q_j^i = \delta_j^i + \dots + [X^i, X^k][X^l, X^m](i_k i_l C^{(6)})_{klmj}. \quad (2.21)$$

However, quadratic couplings of this sort are by no means predicted by T-duality (plus the uplift to M-theory)⁸. Clearly this is due to the fact that an action containing this kind of couplings must be highly non-perturbative. Therefore it could only be derived à la Myers from a non-perturbative action for coincident D-branes. Although such a non-perturbative action is known for a single brane (it is well-known that worldvolume duality of the BI vector yields the action which is valid in the strong coupling regime) it is not known for coincident branes. Therefore, we seem to be stuck with some fundamental problem in D-brane actions.

In the next two sections we show how the action (2.1) can be used to correctly describe microscopically the giant graviton in $AdS_4 \times S^7$ and the dual giant graviton in $AdS_7 \times S^4$. We will compare in both cases to the corresponding macroscopical descriptions and see that there is perfect agreement for large number of gravitons.

3 The giant graviton in $AdS_4 \times S^7$

The giant graviton solution of [7] in this spacetime is in terms of an M5-brane with the topology of a 5-sphere contained in S^7 , carrying angular momentum along the ϕ -direction and magnetic moment with respect to the 6-form potential of the background:

$$C_{\phi\beta_1\dots\beta_5}^{(6)} = -L^6 \sin^6 \theta \sqrt{g_\beta} \quad (3.1)$$

where β_i are the angles parametrising the unit 5-sphere in S^7 :

$$d\Omega_5^2 = d\beta_1^2 + \sin^2 \beta_1 (d\beta_2^2 + \sin^2 \beta_2 (d\beta_3^2 + \sin^2 \beta_3 (d\beta_4^2 + \sin^2 \beta_4 d\beta_5^2))) \quad (3.2)$$

Taking the ansatz $r = 0$, $\theta = \text{constant}$, $\phi = \phi(\tau)$ in (2.7) we expect to find the gravitons expanding into a non-commutative S^5 with radius $L \sin \theta$. As we have mentioned in the previous section, describing the S^5 as a $U(1)$ bundle over the two dimensional complex projective space, CP^2 , it is natural to identify the isometry of the action for the coincident gravitons with the $U(1)$ coordinate.

3.1 S^5 as a $U(1)$ -bundle over CP^2

It is well-known that the 5-sphere can be described as a $U(1)$ bundle over the two dimensional complex projective space. Here we will review some details of this construction and introduce a convenient set of adapted coordinates to the $U(1)$ isometry. We will mainly follow references [28, 29]. The reader is referred to those references for more details.

The unit S^5 can be represented as a submanifold of \mathbb{C}^3 with coordinates (z_0, z_1, z_2) satisfying $\bar{z}_0 z_0 + \bar{z}_1 z_1 + \bar{z}_2 z_2 = 1$. This is invariant under $z_i \rightarrow z_i e^{i\alpha}$, and CP^2 is the space of orbits under the action of this circle group. The projection of points in S^5 onto these orbits is the $U(1)$ -fibration of S^5 . Setting:

$$\xi_1 = z_1/z_0, \quad \xi_2 = z_2/z_0, \quad z_0 = |z_0|e^{i\chi}, \quad (3.3)$$

⁸This is not the case for the Type IIB waves, where the corresponding coupling is a dipole coupling

$$Q_j^i = \delta_j^i + \dots - i[X^i, X^k](i_k i_l C^{(4)})_{kj}$$

neatly predicted by T-duality from the action for Type IIA waves (see [5] for the details).

and defining:

$$A = \frac{i}{2}(1 + |\xi_1|^2 + |\xi_2|^2)^{-1}[\bar{\xi}_1 d\xi_1 + \bar{\xi}_2 d\xi_2 - \text{c.c.}], \quad (3.4)$$

the metric on the S^5 may be written as

$$\begin{aligned} d\Omega_5^2 &= (d\chi - A)^2 + (1 + \sum_k |\xi_k|^2)^{-1} \sum_i |d\xi_i|^2 - \\ &\quad -(1 + \sum_k |\xi_k|^2)^{-2} \sum_{i,j} \xi_i \bar{\xi}_j d\bar{\xi}_i d\xi_j \\ &= (d\chi - A)^2 + ds_{CP^2}^2, \end{aligned} \quad (3.5)$$

since CP^2 is the projection orthogonal to the vector $\partial/\partial\chi$. $ds_{CP^2}^2$ is the Fubini-Study metric for CP^2 :

$$\begin{aligned} ds_{CP^2}^2 &= (1 + \sum_k |\xi_k|^2)^{-1} \sum_i |d\xi_i|^2 - (1 + \sum_k |\xi_k|^2)^{-2} \sum_{i,j} \xi_i \bar{\xi}_j d\bar{\xi}_i d\xi_j = \\ &= \frac{\partial^2 K}{\partial \xi^i \partial \bar{\xi}^j} d\xi^i d\bar{\xi}^j; \end{aligned} \quad (3.6)$$

with $K = \log(1 + |\xi^1|^2 + |\xi^2|^2)$. Therefore CP^2 has a Kähler structure with Kähler form $J = i\partial\bar{\partial}K$. The field strength $F = dA = 2J$ is a solution of Maxwell's equations, the so-called "electromagnetic instanton" of [30]. It is self-dual and satisfies that its integral $\int F \wedge F$ associated with the second Chern class is equal to $4\pi^2$. This solution to Maxwell's equations will in fact play a role in the macroscopical description of giant gravitons of sections 3.4 and 4.1.

One can obtain a real four dimensional metric on CP^2 by defining coordinates $(\varphi_1, \varphi_2, \psi, \varphi_3)$, $0 \leq \varphi_1 \leq \pi/2$, $0 \leq \varphi_2 \leq \pi$, $0 \leq \psi \leq 4\pi$, $0 \leq \varphi_3 \leq 2\pi$, as [29]:

$$\begin{aligned} \xi_1 &= \tan \varphi_1 \cos \frac{\varphi_2}{2} e^{i(\psi+\varphi_3)/2} \\ \xi_2 &= \tan \varphi_1 \sin \frac{\varphi_2}{2} e^{i(\psi-\varphi_3)/2} \end{aligned} \quad (3.7)$$

to give

$$ds_{CP^2}^2 = d\varphi_1^2 + \frac{1}{4} \sin^2 \varphi_1 [\cos^2 \varphi_1 (d\psi + \cos \varphi_2 d\varphi_3)^2 + d\varphi_2^2 + \sin^2 \varphi_2 d\varphi_3^2]. \quad (3.8)$$

In these coordinates the connection A defined in (3.4) is given by

$$A = -\frac{1}{2} \sin^2 \varphi_1 (d\psi + \cos \varphi_2 d\varphi_3) \quad (3.9)$$

and the 6-form potential of the background reads:

$$C_{\phi\chi\varphi_1\varphi_2\psi\varphi_3}^{(6)} = -\frac{1}{8} L^6 \sin^6 \theta \sin^3 \varphi_1 \sin \varphi_2 \cos \varphi_1. \quad (3.10)$$

Clearly, the background is isometric in the χ -direction, and this is the direction that we are going to identify with the isometric direction in the action (2.1), i.e. $k^\mu = \delta_\chi^\mu$.

Let us now make the non-commutative ansatz for the 5-sphere. Inspired by the results of [5] for $AdS_5 \times S^5$, where it was found that the non-commutative manifold on which the giant (and dual giant) graviton expands is defined as an S^1 bundle over a non-commutative S^2 , we make the ansatz that the non-commutative manifold onto which the giant graviton in $AdS_4 \times S^7$ expands is defined as an S^1 bundle over a non-commutative CP^2 .

3.2 The fuzzy CP^2

In this subsection we review some basic properties about the fuzzy CP^2 . The fuzzy CP^2 has been extensively studied in the literature (see for instance [23],[15]-[22]). In this section we will mainly follow the notation in [17].

CP^2 is the coset manifold $SU(3)/U(2)$, and can be defined as the submanifold of \mathbb{R}^8 determined by the constraints:

$$\begin{aligned} \sum_{i=1}^8 x^i x^i &= 1 \\ d^{ijk} x^j x^k &= \frac{1}{\sqrt{3}} x^i \end{aligned} \quad (3.11)$$

where d^{ijk} are the components of the totally symmetric $SU(3)$ -invariant tensor defined by

$$\lambda^i \lambda^j = \frac{2}{3} \delta^{ij} + (d^{ijk} + i f^{ijk}) \lambda^k, \quad (3.12)$$

where λ^i , $i = 1, \dots, 8$ are the Gell-Mann matrices. In this set of constraints only four are independent (the first one, for instance, is a consequence of the rest), therefore they define a four dimensional manifold.

A matrix level definition of the fuzzy CP^2 can be obtained by imposing the conditions (3.11) at the level of matrices. Defining a set of coordinates X^i , $i = 1, \dots, 8$ as

$$X^i = \frac{1}{\sqrt{C_N}} T^i \quad (3.13)$$

with T^i the generators of $SU(3)$ in an N dimensional irreducible representation and C_N the quadratic Casimir of $SU(3)$ in this representation, the first constraint in (3.11) is trivially satisfied through the quadratic Casimir of the group

$$\sum_{i=1}^8 X^i X^i = \frac{1}{C_N} \sum_{i=1}^8 T^i T^i = \frac{1}{C_N} C_N = \mathbb{1}, \quad (3.14)$$

whereas the rest of the constraints are satisfied for any n (see Appendix A in [17]) if the X^i 's are taken in the $(n, 0)$ or $(0, n)$ representations of $SU(3)$, parametrising the irreducible representations of $SU(3)$ by two integers (n, m) corresponding to the number of fundamental and anti-fundamental indices. They become at the level of matrices

$$d^{ijk} X^j X^k = \frac{\frac{n}{3} + \frac{1}{2}}{\sqrt{\frac{1}{3} n^2 + n}} X^i. \quad (3.15)$$

Therefore, to describe the fuzzy CP^2 the non-commuting coordinates X^i have to be taken in the $(n, 0)$ or $(0, n)$ irreducible representations of $SU(3)$. This is in agreement with the fact that G/H cosets can be made fuzzy if H is the isotropy group of the lowest weight state of a given irreducible representation of G [25, 23]. Therefore, different irreducible representations, having associated different isotropy subgroups, can give rise to different cosets G/H . CP^2 has $G = SU(3)$ and $H = U(2)$, and this is precisely the isotropy subgroup of the $SU(3)$

irreducible representations $(n, 0)$ and $(0, n)$. Any other choice of (n, m) has isotropy group $U(1) \times U(1)$, and therefore yields to a different coset, $SU(3)/(U(1) \times U(1))$.

It will be useful later to know that the irreducible representations $(n, 0)$, $(0, n)$ have dimension N given by

$$N = \frac{1}{2}(n+1)(n+2) \quad (3.16)$$

and quadratic Casimir

$$C_N = \frac{1}{3}n^2 + n. \quad (3.17)$$

CP^2 can be embedded in \mathbb{R}^8 using coherent state techniques [31] (see also Appendix B in [23]). In our coordinates (3.8) we have:

$$\begin{aligned} X^1 &= \frac{1}{2} \sin 2\varphi_1 \cos \frac{\varphi_2}{2} \cos \frac{\psi + \varphi_3}{2} \\ X^2 &= -\frac{1}{2} \sin 2\varphi_1 \cos \frac{\varphi_2}{2} \sin \frac{\psi + \varphi_3}{2} \\ X^3 &= \frac{1}{2} [\sin^2 \varphi_1 (1 + \cos^2 \frac{\varphi_2}{2}) - 1] \\ X^4 &= \frac{1}{2} \sin 2\varphi_1 \sin \frac{\varphi_2}{2} \cos \frac{\psi - \varphi_3}{2} \\ X^5 &= -\frac{1}{2} \sin 2\varphi_1 \sin \frac{\varphi_2}{2} \sin \frac{\psi - \varphi_3}{2} \\ X^6 &= \frac{1}{2} \sin^2 \varphi_1 \sin \varphi_2 \cos \varphi_3 \\ X^7 &= -\frac{1}{2} \sin^2 \varphi_1 \sin \varphi_2 \sin \varphi_3 \\ X^8 &= \frac{1}{2\sqrt{3}} (3 \sin^2 \varphi_1 \sin^2 \frac{\varphi_2}{2} - 1), \end{aligned} \quad (3.18)$$

for which

$$\sum_{i=1}^8 (dX^i)^2 = d\varphi_1^2 + \frac{1}{4} \sin^2 \varphi_1 \left[\cos^2 \varphi_1 (d\psi + \cos \varphi_2 d\varphi_3)^2 + d\varphi_2^2 + \sin^2 \varphi_2 d\varphi_3^2 \right] = ds_{CP^2}^2. \quad (3.19)$$

We also have that

$$\sum_{i=1}^8 (X^i)^2 = \frac{1}{3}, \quad (3.20)$$

so that in order to fulfill this constraint we will have to slightly modify our definition (3.13). We then get, in the $(n, 0)$ or $(0, n)$ representations

$$X^i = \frac{1}{\sqrt{3}\sqrt{C_N}} T^i = \frac{1}{\sqrt{n^2 + 3n}} T^i, \quad i = 1, \dots, 8. \quad (3.21)$$

With this normalisation the commutation relations of the X^i become

$$[X^i, X^j] = \frac{i}{\sqrt{n^2 + 3n}} f^{ijk} X^k, \quad (3.22)$$

with f^{ijk} the structure constants of $SU(3)$ in the notation

$$[\lambda^i, \lambda^j] = 2i f^{ijk} \lambda^k. \quad (3.23)$$

3.3 The microscopical description

Let us now take the giant graviton ansatz, $r = 0$, $\theta = \text{constant}$, $\phi = \phi(\tau)$ in the $AdS_4 \times S^7$ background. We find, in Cartesian coordinates

$$ds^2 = -dt^2 + L^2 \cos^2 \theta d\phi^2 + L^2 \sin^2 \theta [(d\chi - A)^2 + (dX^1)^2 + \dots + (dX^8)^2] \quad (3.24)$$

and

$$C_{\phi\chi i j k l}^{(6)} = 2L^6 \sin^6 \theta f^{[ijm} f^{kl]n} X^m X^n. \quad (3.25)$$

Taking now $k^\mu = \delta_\chi^\mu$ in the action (2.1) we have that

$$\begin{aligned} k &= L \sin \theta, & E_{00} &= -1 + L^2 \cos^2 \theta \dot{\phi}^2, \\ Q_j^i &= \delta_j^i - \frac{L^3 \sin^3 \theta}{\sqrt{n^2 + 3n}} f^{ijk} X^k, & i, j &= 1, \dots, 8, \end{aligned} \quad (3.26)$$

and substituting in the action we find

$$S^{\text{BI}} = -T_0 \int d\tau \text{STr} \left\{ \frac{1}{L \sin \theta} \sqrt{1 - L^2 \cos^2 \theta \dot{\phi}^2} \sqrt{1 + \frac{3}{2} \frac{L^6 \sin^6 \theta}{n^2 + 3n} X^2 + \frac{9}{16} \frac{L^{12} \sin^{12} \theta}{(n^2 + 3n)^2} X^2 X^2 + \dots} \right\}. \quad (3.27)$$

Here we have dropped those contributions to $\det Q$ that will vanish when taking the symmetrised trace, and ignored higher powers of $n^2 + 3n$ which will vanish in the large N ($\Leftrightarrow n \rightarrow \infty$) limit. These terms on the other hand cannot be nicely arranged into higher powers of the quadratic Casimir without explicit use of the constraints (3.15). Up to order n^{-4} we have that

$$\sqrt{1 + \frac{3}{2} \frac{L^6 \sin^6 \theta}{n^2 + 3n} X^2 + \frac{9}{16} \frac{L^{12} \sin^{12} \theta}{(n^2 + 3n)^2} X^2 X^2 + \dots} = 1 + \frac{3}{4} \frac{L^6 \sin^6 \theta}{n^2 + 3n} X^2. \quad (3.28)$$

We also have for the CS part of the action:

$$S^{\text{CS}} = \frac{T_0}{2} \int d\tau \text{STr} \left\{ P[(i_X i_X)^2 i_k C^{(6)}] \right\} = \int d\tau \frac{NT_0}{4} \frac{L^6 \sin^6 \theta}{n^2 + 3n} \dot{\phi} \quad (3.29)$$

It is then easy to compute the symmetrised trace to finally arrive, in Hamiltonian formalism, to

$$H = \frac{P_\phi}{L} \sqrt{1 + \tan^2 \theta \left(1 - \frac{NT_0}{4(n^2 + 3n)P_\phi} L^6 \sin^4 \theta \right)^2 + \frac{N^2 T_0^2}{P_\phi^2 \sin^2 \theta} \left(1 + \frac{1}{2} \frac{L^6 \sin^6 \theta}{n^2 + 3n} \right)} \quad (3.30)$$

where P_ϕ is a conserved quantity given that ϕ is cyclic in the Lagrangian. We should stress that this expression is an approximated expression in which higher powers of $n^2 + 3n$ vanishing in the large n limit have been omitted.

In order to describe giant graviton configurations we must set to zero the momentum along the χ -direction, P_χ . Recall however that, by construction, the N gravitons carry momentum $P_\chi = NT_0$. The difference between P_χ being zero or not is merely a coordinate transformation, a boost in χ . However, how to perform coordinate transformations in non-Abelian actions is an open problem [32, 33, 34, 35]. In order to clarify how the limit $P_\chi \rightarrow 0$ must be taken we

study in the next subsection the macroscopical description of this microscopical configuration, in terms of a spherical M5-brane carrying both P_ϕ and P_χ charges. The agreement between the two descriptions in the large N limit will make clear that in the limit $P_\chi \rightarrow 0$ the last term in the Hamiltonian should vanish. On the other hand, the term

$$\frac{NT_0}{4(n^2 + 3n)P_\phi}, \quad (3.31)$$

whose presence is in fact crucial in order to find the giant graviton configuration, remains finite, since $N = (n+1)(n+2)/2$ for $(n,0)$ and $(0,n)$ irreps, so that numerator and denominator in (3.31) scale with the same power of n . In fact, in the $N \rightarrow \infty$ ($\Leftrightarrow n \rightarrow \infty$) limit both terms are compensated, and the finite result $T_0/8P_\phi$ is reached, in perfect agreement with the macroscopical calculation. This compensation does not however take place in the last term in the Hamiltonian.

Therefore, in order to describe giant graviton configurations we minimize

$$H_{\text{mic}} = \frac{P_\phi}{L} \sqrt{1 + \tan^2 \theta \left(1 - \frac{NT_0}{4(n^2 + 3n)P_\phi} L^6 \sin^4 \theta \right)^2} \quad (3.32)$$

with respect to θ . We then find two solutions with energy P_ϕ/L : $\sin \theta = 0$, which corresponds to the point-like graviton, and

$$\sin \theta = \left(\frac{4(n^2 + 3n)P_\phi}{NT_0 L^6} \right)^{1/4} \quad (3.33)$$

which corresponds to the giant graviton, with radius

$$R = \left(\frac{4(n^2 + 3n)P_\phi}{NT_0 L^2} \right)^{1/4}. \quad (3.34)$$

Clearly, for the giant graviton

$$P_\phi \leq \frac{NT_0 L^6}{4(n^2 + 3n)} \quad (3.35)$$

which provides the microscopical bound to the angular momentum predicted by the stringy exclusion principle. When the number of gravitons is large we find that

$$P_\phi \leq \frac{T_0 L^6}{8} = \tilde{N} \quad (3.36)$$

with \tilde{N} the units of 6-form flux on the 5-sphere in the macroscopical description of [7, 8], given by $\tilde{N} = A_5 T_5 L^6$ with A_5 the area of the unit 5-sphere. We therefore find perfect agreement with the bound found in [7]. The same holds true for the radius of the configuration, which for large number of gravitons is given by

$$R = \left(\frac{8P_\phi}{T_0 L^2} \right)^{1/4} = L \left(\frac{P_\phi}{\tilde{N}} \right)^{1/4} \quad (3.37)$$

exactly as in [7, 8].

In this section we have achieved the right microscopical description of the giant graviton in $AdS_4 \times S^7$ in terms of gravitons expanding into a “fuzzy 5-sphere”, defined as a $U(1)$ bundle

over the fuzzy CP^2 . We have seen that the coordinate along the fibre must be isometric in the action, and this has forced our choice of k^μ pointing on that direction, which has in turn introduced, by construction, a non-vanishing momentum P_χ on the configuration. The macroscopical description of such a configuration should then be in terms of a spherical M5-brane carrying both P_ϕ and P_χ charges. We will perform the detailed macroscopical description in these terms in the next subsection. The agreement between the two descriptions will provide further justification to the limit taken to arrive at expression (3.32) as the right microscopical Hamiltonian describing gravitons with angular momentum P_ϕ .

3.4 The macroscopical calculation

The simplest way to describe an M5-brane living in a spacetime with a $U(1)$ isometry and carrying momentum along that direction is by uplifting a D4-brane to M-theory keeping the eleventh direction compact. The resulting brane is therefore a longitudinal 5-brane. Doing this we obtain the action adequate to describe an M5-brane whose worldvolume contains an isometric direction. Clearly this applies to the M5-brane with the topology of an S^5 . In this case the isometric direction is the coordinate along the fibre in the decomposition of the S^5 as a $U(1)$ fibre over CP^2 . The worldvolume of such a brane is therefore effectively four-dimensional, and locally that of a CP^2 .

For the non-vanishing eleven dimensional fields involved in our AdS backgrounds we find the following action for a wrapped M5-brane:

$$S = -T_4 \int d^5\xi \left\{ k\sqrt{\det(\mathcal{G} + k^{-1}F)} - P[i_k C^{(6)}] - \frac{1}{2}P[k^{-2}k^{(1)}] \wedge F \wedge F \right\}. \quad (3.38)$$

Here F is the field strength associated to M2-branes wrapped on the isometric direction ending on the M5-brane, since the uplifting of the BI field strength of the D4-brane, $F + B^{(2)}$, to M-theory gives $F + i_k C^{(3)}$ ⁹. \mathcal{G} is the reduced metric defined in (2.3) and we have denoted T_4 the tension of the brane to explicitly take into account that its spatial worldvolume is 4-dimensional.

As we discussed in Section 2 $k^{-2}k^{(1)}$ is identified with the momentum operator along the isometric direction. Therefore we can switch on momentum charge on the M5-brane by choosing a non-vanishing field strength such that

$$\int_{CP^2} F \wedge F = 8\pi^2 N, \quad (3.39)$$

since then

$$\frac{T_4}{2} \int d\tau d^4\sigma P[k^{-2}k^{(1)}] \wedge F \wedge F = NT_0 \int d\tau P[k^{-2}k^{(1)}]. \quad (3.40)$$

With F satisfying (3.39) we are therefore dissolving N gravitons, propagating along the isometric direction, in the worldvolume of the 5-brane. The field satisfying this condition is in fact the electromagnetic instanton that we discussed in section 3.1, which now must have instanton number equal to twice the number of gravitons. This gives

$$F = \sqrt{\frac{N}{2}} \left(-\sin 2\varphi_1 d\varphi_1 \wedge d\psi - \sin 2\varphi_1 \cos \varphi_2 d\varphi_1 \wedge d\varphi_3 + \sin^2 \varphi_1 \sin \varphi_2 d\varphi_2 \wedge d\varphi_3 \right). \quad (3.41)$$

⁹In the action (3.38) we have set $C^{(3)}$ to zero, which is valid for our particular backgrounds.

Identifying the isometric direction in the action (3.38) with χ in (3.5) and integrating over the spatial worldvolume of the M5-brane we arrive at

$$S = -4\pi^2 T_4 \int d\tau \left\{ L \sin \theta \sqrt{1 - L^2 \cos^2 \theta \dot{\phi}^2} \left(\frac{L^4 \sin^4 \theta}{8} + \frac{N}{L^2 \sin^2 \theta} \right) - \frac{1}{8} L^6 \sin^6 \theta \dot{\phi} \right\} \quad (3.42)$$

and, in Hamiltonian formalism, to

$$H = \frac{P_\phi}{L} \sqrt{1 + \tan^2 \theta \left(1 - \frac{\pi^2 T_4 L^6 \sin^4 \theta}{2 P_\phi} \right)^2 + \frac{16\pi^4 T_4^2 N^2}{P_\phi^2 \sin^2 \theta} \left(1 + \frac{L^6 \sin^6 \theta}{4N} \right)}. \quad (3.43)$$

Comparing with the Hamiltonian constructed in [7, 8], which describes a spherical 5-brane with momentum P_ϕ , we see that a non-vanishing momentum along the χ -direction translates into an additional piece depending on N inside the squared root. Comparing this expression with the microscopical Hamiltonian (3.30) we find that they exactly agree in the large N limit, when $N \sim n^2/2$. In the macroscopical Hamiltonian (3.43) it is clearer however that the limit of zero momentum along the fibre direction is reached when there are zero gravitons propagating in the χ direction dissolved in the worldvolume of the 5-brane, which neatly sets to zero the last term in the Hamiltonian. This further justifies the elimination of the corresponding term in the microscopical Hamiltonian (3.30) in section 3.3.

4 The dual giant graviton in $AdS_7 \times S^4$

4.1 The microscopical description

In this section we briefly describe the microscopical description of the dual giant graviton in $AdS_7 \times S^4$. One expects that microscopically the gravitons expand into a 5-sphere with radius r due to the coupling to the 6-form potential $C_{t\alpha_1\dots\alpha_5}^{(6)}$. As before, we describe the “fuzzy 5-sphere” as a $U(1)$ bundle over the fuzzy CP^2 , and we embed the CP^2 in \mathbb{R}^8 . We then have for the dual giant graviton ansatz:

$$\begin{aligned} ds^2 &= -(1 + \frac{r^2}{\tilde{L}^2}) dt^2 + L^2 d\phi^2 + r^2 \left[(d\chi - A)^2 + (dX^1)^2 + \dots + (dX^8)^2 \right] \\ C_{0\chi i j k l}^{(6)} &= 2 \frac{r^6}{\tilde{L}} f^{[ijm} f^{kl]n} X^m X^n \end{aligned} \quad (4.44)$$

Substituting in the action (2.1) we have that

$$\begin{aligned} k &= r, & E_{00} &= -(1 + \frac{r^2}{\tilde{L}^2}) + L^2 \dot{\phi}^2, \\ Q_j^i &= \delta_j^i - \frac{r^3}{\sqrt{n^2 + 3n}} f^{ijk} X^k, & i, j &= 1, \dots, 8 \end{aligned} \quad (4.45)$$

and

$$\begin{aligned} S &= -T_0 \int d\tau \text{STr} \left\{ \frac{1}{r} \sqrt{1 + \frac{r^2}{\tilde{L}^2} - L^2 \dot{\phi}^2} \sqrt{1 + \frac{3}{2} \frac{r^6}{n^2 + 3n} X^2 + \frac{9}{16} \frac{r^{12}}{(n^2 + 3n)^2} X^2 X^2} + \dots \right. \\ &\quad \left. - \frac{r^6}{\tilde{L}} \frac{N}{4(n^2 + 3n)} \right\}. \end{aligned} \quad (4.46)$$

Up to order n^{-4} we arrive at the Hamiltonian

$$H = \sqrt{\left(1 + \frac{r^2}{\tilde{L}^2}\right)\left(\frac{P_\phi^2}{L^2} + \frac{N^2 T_0^2 r^{10}}{16(n^2 + 3n)^2} + \frac{N^2 T_0^2}{r^2}(1 + \frac{1}{2} \frac{r^6}{n^2 + 3n})\right)} - \frac{NT_0}{4(n^2 + 3n)} \frac{r^6}{\tilde{L}} \quad (4.47)$$

Now we have to take the limit $P_\chi = 0$ which amounts to setting to zero the last term in the squared-root, as we discussed in detail when we studied the giant graviton in $AdS_4 \times S^7$. We are then left with

$$H_{\text{mic}} = \sqrt{\left(1 + \frac{r^2}{\tilde{L}^2}\right)\left(\frac{P_\phi^2}{L^2} + \frac{N^2 T_0^2 r^{10}}{16(n^2 + 3n)^2}\right)} - \frac{NT_0}{4(n^2 + 3n)} \frac{r^6}{\tilde{L}}, \quad (4.48)$$

which again is an approximated expression in which higher order powers of $n^2 + 3n$ vanishing in the large n limit have been omitted. Minimizing with respect to r we find two solutions with energy P_ϕ/L : $r = 0$, which corresponds to the point-like graviton, and

$$r = \left(\frac{4(n^2 + 3n)P_\phi}{NT_0 L \tilde{L}}\right)^{1/4} \quad (4.49)$$

which corresponds to the dual giant graviton. When $N \rightarrow \infty$

$$r \rightarrow \left(\frac{8P_\phi}{T_0 L \tilde{L}}\right)^{1/4} = \tilde{L} \left(\frac{P_\phi}{\tilde{N}}\right)^{1/4} \quad (4.50)$$

in agreement with the result in [8].

4.2 The macroscopical calculation

Using the action (3.38) we can easily describe the previous configurations in terms of an M5-brane wrapped on the χ -direction and carrying P_ϕ and P_χ momentum charges. P_χ is switched on by dissolving N gravitons with momentum on this direction in the worldvolume of the M5-brane. We find

$$S = -4\pi^2 T_4 \int d\tau \left[r \left(\frac{r^4}{8} + \frac{N}{r^2} \right) \sqrt{1 + \frac{r^2}{\tilde{L}^2} - L^2 \dot{\phi}^2} - \frac{1}{8} \frac{r^6}{\tilde{L}} \right], \quad (4.51)$$

and, in Hamiltonian formalism

$$H = \sqrt{1 + \frac{r^2}{\tilde{L}^2}} \sqrt{\frac{P_\phi^2}{L^2} + 16\pi^4 T_4^2 r^2 \left(\frac{r^4}{8} + \frac{N}{r^2}\right)^2} - \frac{\pi^2 T_4}{2} \frac{r^6}{\tilde{L}}. \quad (4.52)$$

Clearly the condition $P_\chi = 0$ is met for $N = 0$ gravitons dissolved in the worldvolume, which amounts to setting to zero the N -term in the squared-root. Comparing to (4.47) this further justifies the limit taken in that expression, yielding to the microscopical potential (4.48). In fact, when $N \rightarrow \infty$ ($\Leftrightarrow n \rightarrow \infty$) we find perfect agreement between (4.52) and the microscopical potential (4.48).

5 Conclusions

We have shown that the giant graviton configurations in $AdS_m \times S^n$ backgrounds that involve 5-spheres are described microscopically in terms of gravitational waves expanding into “fuzzy 5-spheres” which are defined as S^1 bundles over fuzzy CP^2 . The explicit construction can be done due to the fact that the action used to describe the system of coincident waves contains a special $U(1)$ isometric direction that can be identified with the $U(1)$ fibre.

In this description the gravitons expand into a longitudinal M5-brane which has four manifest dimensions and one wrapped on the $U(1)$ direction. This brane carries quadrupolar magnetic moment with respect to the 6-form potential of the background in the $AdS_4 \times S^7$ case, or quadrupolar electric moment in the $AdS_7 \times S^4$ case. In both cases it has a non-vanishing angular momentum along the spherical part of the geometry, P_ϕ . The details of the construction show that there is as well a non-vanishing momentum along the $U(1)$ direction, which has to be set to zero to find the right point-like graviton and giant graviton configurations in the background. We have seen that in that case not only the radii of the giant gravitons but also the Hamiltonian that they minimize agree with the macroscopical results in [7, 8] for large number of gravitons.

Gravitational waves propagating both along the spherical part of the geometry (the ϕ direction) and the compact $U(1)$ direction can be described macroscopically in terms of a longitudinal M5-brane with velocity $\dot{\phi}$ wrapped on the isometric direction. The action associated to this brane can be easily constructed by just uplifting the action of the D4-brane to M-theory, while maintaining the eleventh direction compact. The $\int C^{(1)} F \wedge F$ term in the CS part of the D4-brane action is then uplifted to $\int k^{-2} k^{(1)} F \wedge F$, with F now associated to wrapped M2-branes ending on the M5-brane. Therefore a momentum charge along the compact direction is simply switched on by taking F with non-vanishing instanton number. The comparison between this description and our microscopical description shows exact agreement for large number of gravitons. Moreover, this comparison can be used to clarify the right way to set to zero the momentum along the compact direction to finally obtain the correct giant graviton configurations.

Our action for M-theory waves, therefore, provides an explicit Matrix action which is solved by some sort of non-commutative 5-sphere. Moreover, although we have not checked the supersymmetry properties of our configurations, the agreement with the macroscopical description of [7, 8] suggests that they should occur as BPS solutions preserving the same half of the supersymmetries as the point-like graviton [8]. To our knowledge this would be the first example of a physical matrix model, coming up as the action for coincident M-theory gravitational waves, admitting some fuzzy 5-sphere as a supersymmetry preserving solution.

Let us stress that the “fuzzy 5-sphere” that we have constructed is defined as an S^1 bundle over the fuzzy CP^2 , and is therefore different from previous fuzzy 5-spheres discussed in the literature [12, 13, 14]. In particular, our solution does not show $SO(6)$ covariance, this invariance being broken down to $SU(3) \times U(1)$, whereas this is the case for the fuzzy 5-sphere in [12, 13, 14]. The $SO(6)$ invariance might still be present in a non-manifest way, after all the $SO(6)$ covariance of the classical 5-sphere is also not explicit when it is described as an S^1 -bundle over CP^2 . Another difference is that our solution approaches neatly the classical S^5 in the large N limit, where all the non-commutativity disappears. This is not the case for the fuzzy 5-sphere in [12, 13, 14]. Indeed, the right dependence of the radius of the 5-sphere giant graviton with P_ϕ , $R \sim P_\phi^{1/4}$, is only achieved within the S^1 bundle over CP^2 description, the corresponding dependence of the fuzzy 5-sphere of [12, 13, 14] being given by

$R \sim P_\phi^{1/5}$. Another difference is that our “fuzzy S^5 ” inherits its symplectic structure from the Kähler form of the fuzzy CP^2 , whereas in the construction in [12, 13] the bundle structure corresponds to a CP^3 base and a CP^2 fibre. Therefore, there are clear differences between the two constructions.

Non-supersymmetric longitudinal M5-branes with $CP^2 \times S^1$ topology have been obtained as explicit solutions of Matrix theory in [15]. Our longitudinal M5-branes, although similar in the explicit construction, have S^5 topology, once the necessary twist in the fibre is taken into account. This twist should provide the global extension of the local residual supersymmetry found in [15], in terms of spinors charged under the gauge potential whose field strength is the Kähler form (see [36, 29]).

Longitudinal 5-branes with other topologies have also been shown to arise as solutions to Matrix theory in [37, 38, 39, 40] (see also the fuzzy funnel solution in [41]). In general, to find these solutions it is necessary to include additional Chern-Simons terms or mass terms. Very recently [42] there has been some speculation on how the fuzzy 5-sphere of [12, 13] might appear as a solution to the pp-wave Matrix model of [3]. It would be interesting to elucidate the relation between the new Chern-Simons coupling conjectured in this reference and the dielectric couplings constructed in this paper once the Penrose limit is taken. This would help clarifying if indeed the resulting Matrix action would allow for transverse 5-brane solutions [43].

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